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Natural Frequencies of a Cantilever with an Asymmetrically Attached Tip Mass

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BHAT and co-workers[1,2] have derived the exact frequency equation for a uniform cantilever beam carrying a tip mass that is slender in the axial direction. The centroid of the tip mass was assumed to lie on the extended neutral axis of the beam when it is in its undeformed configuration. The significant distinct feature of those investigations was that the centroid of the tip mass did not coincide with its point of attachment to the beam. In view of the importance of this problem in airplane and missile design, it is of interest to consider a closely related problem, namely, that in which the centroid of the tip mass does not coincide with its point of attachment to the beam and in which it is also offset an arbitrary distance perpendicular to the extended neutral axis of the beam. In this case, the longitudinal and transverse deflections in the beam become coupled through the boundary conditions because of the presence of the asymmetrically oriented tip mass (see Fig. 1).

Consider a slender elastic cantilever of length ℓ and density ρ carrying a rigid tip mass m that is undergoing longitudinal and flexural deflections $u(x_1,t)$ and $w(x_1,t)$, respectively. Following a derivation similar to that presented in Ref. 3, one can show that the equations of motion for small deflections are

$$\left.\begin{aligned} EAu,_{11} &= \rho A\ddot{u}, \\ EIw,_{1111} + \rho A\ddot{w} &= 0, \end{aligned}\right\} 0<x_1<\ell,\ 0<t \qquad (1)$$

where $u,_1 = \partial u/\partial x_1$, $\dot{u} = \partial u/\partial t$, etc., E denotes Young's modulus for the beam, A its constant cross-sectional area, and I its moment of inertia. The boundary conditions are

$$u = w = w,_1 = 0 \qquad (2)$$

at the clamped end $x_1 = 0$ and

$$EAu,_1 + m\ddot{u} - mr_1\ddot{w},_1 = 0 \qquad (3)$$

$$EIw,_{11} + (mc^2 + J)\ddot{w},_1 + mc\ddot{w} - mr_1\ddot{u} = 0 \qquad (4)$$

$$EIw,_{111} - m\ddot{w} - mc\ddot{w},_1 = 0 \qquad (5)$$

at the free end $x_1 = \ell$, where r_1 is the distance from o to C (see Fig. 1), C being the centroid of the tip mass, c is the half thickness of the tip mass (assumed here to be symmetric about the vertical axis through o and C), and J is its moment of inertia, with $J = ma_0^2$, a_0 being its radius of gyration.

For free harmonic oscillation of dimensionless natural frequency ω, one may assume that

$$u(x_1,t) = u(x)\cos\omega\tau, \quad w(x_1,t) = w(x)\cos\omega\tau$$

so that Eqs. (1-5) can be expressed in dimensionless form as

$$u''(x) + \omega^2 u(x) = 0, \quad w^{IV}(x) - \lambda^4 w(x) = 0, \quad 0<x<1 \qquad (6)$$

$$u(0) = w(0) = w'(0) = 0 \qquad (7)$$

$$u'(1) - \mu\omega^2 u(1) + \mu r\omega^2 w'(1) = 0 \qquad (8)$$

$$w''(1) - \mu(\xi^2 + a^2)\lambda^4 w'(1) - \mu\xi\lambda^4 w(1) + \mu r\lambda^4 u(1) = 0 \qquad (9)$$

$$w'''(1) + \mu\lambda^4 w(1) + \mu\xi\lambda^4 w'(1) = 0 \qquad (10)$$

where $u' = \mathrm{d}u/\mathrm{d}x$, etc., and where the following variables and dimensionless parameters have been introduced:

$$x = x_1/\ell, \quad \tau = t/\ell(\rho/E)^{1/2}, \quad r = r_1/\ell, \quad \xi = c/\ell$$

$$\mu = m/\rho A\ell, \quad \alpha^2 = I/A\ell^2, \quad \lambda^2 = \omega/\alpha, \quad a = a_0/\ell \qquad (11)$$

Solving the eigenvalue problem in Eqs. (6-10), one obtains the following frequency equation:

$$\begin{aligned} &(\cos\omega - \mu\omega\sin\omega)\{1 + (\mu a\lambda^2)^2 + [1 - (\mu a\lambda^2)]\cos\lambda\cosh\lambda \\ &+ \mu\lambda[1 - \lambda^2(\xi^2 + a^2)]\cos\lambda\sinh\lambda - \mu\lambda[1 + \lambda^2(\xi^2 + a^2)] \\ &\times \sin\lambda\cosh\lambda - 2\mu\xi\lambda^2\sin\lambda\sinh\lambda\} + (\mu r)^2\omega\lambda^3\sin\omega[\mu\lambda(1 \\ &- \cos\lambda\cosh\lambda) - \cos\lambda\sinh\lambda - \sin\lambda\cosh\lambda] = 0 \end{aligned} \qquad (12)$$

In the event that the dimensionless coupling parameter r vanishes, Eq. (12) reduces to two separate frequency equations, namely, that reported in Refs. 1 and 2 for the free flexural vibrations of a cantilever with a slender tip mass and that for the free extension vibrations of a fixed-free bar carrying a tip mass at its free end, i.e., $\cot\omega = \mu\omega$.

For the purpose of determining numerically the natural frequencies ω_n $(n = 1,2,3,\ldots)$ from Eq. (12), it is necessary to specify the geometry of the beam and the tip mass. Suppose, for the sake of example, that the beam has a rectangular cross section of dimensions $2b_1 \cdot h$, so that $A = 2b_1 h$ and $I = 2hb_1^3/3$. Let the tip mass be a rectangular parallelopiped of length ℓ_0, thickness $2c$, and depth h with mass $m = 2\rho_0 ch\ell_0$, where ρ_0 is the density of the tip mass material. Moreover, consider the special case of $|oP| = b_1$ (see Fig. 1). Then

$$\alpha = \frac{b_1}{\sqrt{3}}, \quad \mu = \frac{\rho_0 c\ell_0}{\rho b_1 \ell}, \quad r = \frac{(\ell_0 - 2b_1)}{2\ell}$$

$$a = \frac{1}{\sqrt{3}}(c^2 + \ell_0^2 - 3b_1\ell_0 + 3b_1^2)$$

For specific numerical values of some of the parameters, suppose that $\rho_0/\rho = 2.945$, $b_1 = 0.1$ cm, $c = 0.2$ cm, and $\ell = 5$ cm, while the value of ℓ_0 shall be varied, which, then, implies that the tip mass, the centroid, and the radius of inertia parameters μ, r, and a, respectively, vary as well. In this case, $\alpha = 0.01155$.

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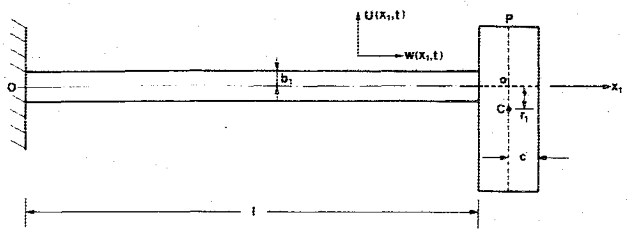

Fig. 1 Coordinate system and dimensions.

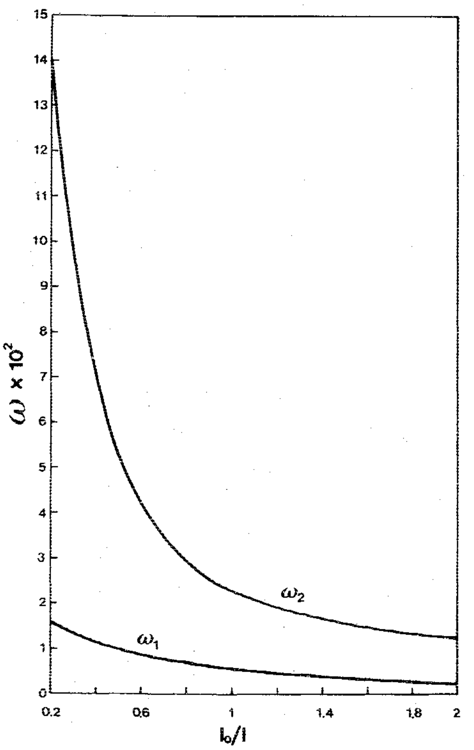

Fig. 2 Variation of ω_1, ω_2 with ℓ_0/ℓ for $\rho_0/\rho = 2.945$, $b_1 = 0.1$ cm, $c = 0.2$ cm, and $\ell = 5$ cm.

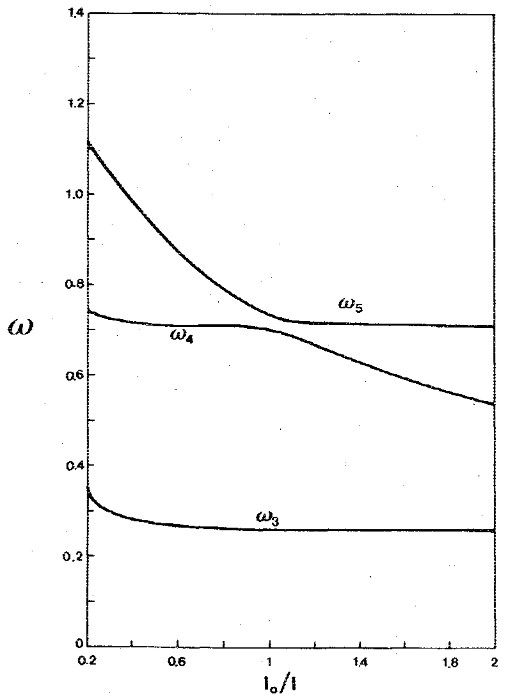

Fig. 3 Variation of ω_3, ω_4, ω_5 with ℓ_0/ℓ for $\rho_0/\rho = 2.945$, $b_1 = 0.1$ cm, $c = 0.2$ cm, and $\ell = 5$ cm.

Equation (12) has been solved numerically for the first five natural frequencies for the numerical values of the parameters given earlier and over the interval $0.2 \leq \ell_0/\ell \leq 2.0$. The results are shown in Figs. 2 and 3. The variations of ω_1 and ω_2 with ℓ_0/ℓ are plotted in Fig. 2, from which it is evident that both frequencies decrease monotonically as ℓ_0/ℓ increases. The value of the second frequency initially decreases quite rapidly, but for $\ell_0/\ell > 1$ its rate of decrease is considerably diminished. From Fig. 3, it is evident that the dimensionless frequencies ω_3, ω_4, and ω_5 also decrease monotonically as ℓ_0/ℓ increases. Initially, ω_3 decreases slightly and thereafter remains virtually constant. On the interval $0.2 \leq \ell_0/\ell \leq 1.0$, ω_4 decreases very slowly, whereas ω_5 decreases noticeably and almost linearly. Near $\ell_0/\ell = 1$, the values of ω_4 and ω_5 differ by about 4%, but for $1 < \ell_0/\ell < 2$, the roles are reversed, with ω_5 decreasing only a very small amount and ω_4 decreasing rather sharply in an almost linear manner.

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Entrainment Characteristics of Unsteady Subsonic Jets

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Introduction

THE entrainment mechanism in turbulent jets has been a subject of considerable basic and applied interest for many years. Recently, this problem has received increased attention because of the need to develop compact, yet highly efficient thrust augmenting ejectors for VSTOL applications.[1] Several new techniques have been introduced or proposed to increase the jet entrainment, e.g., hypermixing,[2] swirling,[3] acoustic interaction,[4] and unsteady jet techniques.[5] It is the objective of this paper to present recent results on the entrainment characteristics of two types of unsteady jet flows, i.e., oscillating jets with time-varying jet deflection and pulsating jets with time-varying mass flow.

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